Rice University
Jones Graduate School of Business

BUSI 573: Stochastic Models in Operations Management
PhD Qualification Exam
May 21, 2026

Instructions.

1. This is an open lecture notes exam.
2. No electronics are allowed.
3. The exam has 3 problems. Please make sure that you have all the pages.
4. Please show all your work.
5. You have 180 minutes.

In taking this examination, I acknowledge and accept the Rice University Honor Code. On my honor, I have neither given nor received any unauthorized aid on this exam.

Name:

Signature:

GOOD LUCK!

Problem 1	Problem 2	Problem 3	Total

Problem 1. (30 points) Throughout this problem, you will work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. You may freely cite the Monotone Convergence Theorem (Theorem 1.11 in the notes), Fatou's Lemma (Proposition 1.12 in the notes), and Lebesgue's Dominated Convergence Theorem (Theorem 1.13 in the notes).

Let $U \sim U[0,1]$, and for each integer $n \geq 1$, define

$$X_n := n^2 \cdot \mathbb{1}\left\{\frac{1}{n+1} < U \leq \frac{1}{n}\right\}.$$

a. Prove that $X_n \to_{\text{a.s.}} 0$.

b. Compute $\lim_{n\to\infty} \mathbb{E}[X_n]$ and deduce that the interchange of expectation and limit fails.

c. Determine whether the hypotheses of the Monotone Convergence Theorem and Fatou's Lemma are satisfied by $(X_n)_{n\geq 1}$. If the hypotheses are satisfied, state explicitly what the conclusion is.

d. Show that there is no non-negative random variable Y with $\mathbb{E}[Y] < \infty$ and $X_n \leq Y$ almost surely for all $n \geq 1$. *Hint. Assume to the contrary that such a Y exists. Then consider what such a Y would imply for* $\mathbb{E}[1/(U^2)]$.

e. A sequence of random variables $(Y_n)_{n\geq 1}$ is said to be *uniformly integrable* if

$$\lim_{K\to\infty} \sup_{n\geq 1} \mathbb{E}[|Y_n| \; \mathbb{1}\{|Y_n| > K\}] = 0.$$

Prove or disprove that $(X_n)_{n\geq 1}$ is uniformly integrable.

Problem 2. (30 points) Define the moment generating function of a random variable X as $M_X(\lambda) := \mathbb{E}[e^{\lambda X}]$ for $\lambda \in \mathbb{R}$. Then the Chernoff method states that for any random variable X, for any $a \in \mathbb{R}$, and for any $\lambda > 0$, we have

$$\mathbb{P}(X \geq a) \leq e^{-\lambda a} M_X(\lambda),$$

where the Chernoff bound is optimized by choosing the optimal λ that minimizes the right-hand side. Let $(G_k)_{1 \leq k \leq n}$ be a sequence of independent geometric random variables with $G_k \sim \text{Geometric}(p_k)$ for all $1 \leq k \leq n$. Let $T_n = \sum_{k=1}^n G_k$, $H_n = \sum_{k=1}^n \frac{1}{k}$, and $p_k = \frac{k}{n}$ for all $1 \leq k \leq n$.

a. Using the sum formula of geometric series, i.e., $a + ar + ar^2 + \cdots = \frac{a}{1-r}$ for $|r| < 1$, prove that for all $1 \leq k \leq n$, we have

$$M_{G_k}(\lambda) = \frac{p_k e^\lambda}{1 - (1 - p_k)e^\lambda} \qquad \text{for} \qquad \lambda < -\log(1 - p_k).$$

b. Recall that $p_k = \frac{k}{n}$ for all $1 \leq k \leq n$. Using the fact that $\log(M_{G_k}(\lambda)) \leq \frac{\lambda}{p_k} + \frac{2\lambda^2}{p_k^2}$, show that

$$\log \mathbb{E}[e^{\lambda T_n}] \leq \lambda n H_n + 2\lambda^2 n^2 \sum_{j=1}^n \frac{1}{j^2} \qquad \text{for } \lambda \in (0, \frac{1}{2n}].$$

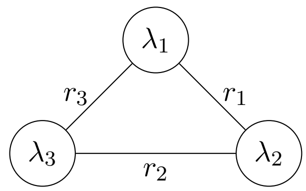
c. Take $\lambda = \frac{1}{2n}$. Apply the Chernoff method to upper bound $\mathbb{P}(T_n \geq nH_n + cn)$ for all $c > 0$.

d. Let $M_n := \max_{1 \leq k \leq n} G_k$. Prove that for every $c > 0$,

$$\limsup_{n \to \infty} \mathbb{P}(M_n \geq cn) \leq \frac{e^{-c}}{1 - e^{-c}}.$$

Hint. Compute $\mathbb{P}(G_k \geq t)$ explicitly for all $t \in \mathbb{N}$ and use the union bound.

Problem 3. (40 points) Throughout this problem, you will work on a dynamic matching problem on the triangle network (an odd cycle of length 3) $\mathcal{G}$ below. Fix a small parameter $\epsilon \in (0, \frac{1}{4})$, and let $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (\frac{1}{2} + \epsilon, \frac{1}{4} - \frac{\epsilon}{2}, \frac{1}{4} - \frac{\epsilon}{2})$ and $r = (r_1, r_2, r_3) = (1, 1, 1)$. You may freely cite the hindsight optimality test lemma (Lemma 5.5 in the notes) and bounding stationary expectations lemma (Lemma 5.6 in the notes).



a. Solve the static planning problem, and verify that the general position condition holds. Identify z^*, s^*, and the sets $\mathcal{M}_+, \mathcal{M}_0, \mathcal{Q}_+, \mathcal{Q}_0$ (see Definition 5.3 in the notes).

b. Consider the following greedy matching policy D: whenever possible, perform a match in $\mathcal{M}_+$ (if multiple matches are available at the same time, break ties uniformly at random), and never perform a match in $\mathcal{M}_0$. Show that $\mathbb{E}[Q_1^t] \to \infty$ as $t \to \infty$.

c. Consider the Lyapunov function $g(Q^t) := (Q_2^t)^2 + (Q_3^t)^2$ on the state space $(Q_1^t, Q_2^t, Q_3^t) \in \mathbb{Z}_{\geq 0}^3$. We want to compute an upper bound on the one-step drift

$$\Delta g(Q^t) := \mathbb{E}[g(Q^{t+1}) - g(Q^t) \mid Q^t]$$

using the following worst-case assumption: any arrival to queue-1 leaves the system unmatched if queues 2 and 3 are empty. Compute the upper bound in each of the four possible cases: $\{Q_2^t = 0, Q_3^t = 0\}, \{Q_2^t = 0, Q_3^t \geq 1\}, \{Q_2^t \geq 1, Q_3^t = 0\}$, and $\{Q_2^t \geq 1, Q_3^t \geq 1\}$.

d. By combining the four cases, give a single bound of the form

$$\Delta g(Q^t) \leq -\delta(Q_2^t + Q_3^t) + \Gamma \quad \text{for all} \quad (Q_1^t, Q_2^t, Q_3^t) \in \mathbb{Z}_{\geq 0}^3,$$

where δ, Γ are some positive constants.

e. Prove that D is hindsight optimal and $\mathcal{R}^{*,t} - \mathcal{R}^{D,t} = O(\epsilon^{-1})$.